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Dynamic Model of Optimal Production Control in a Hysteretic Behaviour of Economic Agents

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Abstract:

The hysteresis nature of demand, depending on the price ratio and purchasing power, is well-known, but mathematical models that describe this relationship have not been developed so far.

This article provides a dynamic model of optimal production within price hysteretic behaviour. Unlike the standard description methods, the proposed cobweb model provides for qualitative description of the solution to the equation at the slowly changing control. The article also describes the production model enable to determine the adjustment dynamics of product quantity from the manufacturer and the consumer. The solution is described by standard methods of optimal control theory.

Keywords: *Hysteretic pricing model, Hysteretic price behaviour, Optimal manufacturing strategy, Demand function.*

JEL Classification: *A10, C10, M11, M21*

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1. Introduction

In the economics, the presence of hysteresis phenomena has been observed since the 1950s. However, in the economy papers with the formal description within the framework of hysteresis phenomena systems theory have emerged only in the recent decades. The work considers a significant difference of the properties of systems with hysteresis from systems with functional non-linearities. This can be explained by the complexity and non-linear structure of the states of hysteresis quantizer space.

Appearance of the hysteretic effects is observed in the economy at different levels. In the middle of the last century, economists identified hysteresis at the micro level with the sustainability of consumer habits. Concerning the macro level, it is possible to observe the hysteretic effect at increasing unemployment when affected by the certain motivating factors. Disappearance of these factors does not lead to unemployment slowdown, and it stays at a sufficiently high level for a long time.

One of the main problems of mathematical modelling in the economy is the problem of studying of the process for establishing equilibrium price. In the classic sense, cobweb model and its analogues are used when speaking about the functions of supply and demand in the context of the pricing question. In the course of modern researches it is shown that the status of the economic system at some point in time depends on the parameters values both in the current and the previous point of time. From this follows the need to develop a mathematical model of the demand function, taking into account this feature. The hysteresis quantizers are the best way to deal with the problem.

2. Literature Review

Formal description of the hysteresis quantizers is based on the operator representation of converters developed by Krasnosel'skii (Krasnosel'skii and Pokrovskii, 1989) and his followers (Semenov and Meleshenko *et al,* 2014; Semenov, Solovyov *et al.,* 2015; Thalassinos et al., 2009; 2012a; 2012b; 2015, Hanias *et al.,* 2007). They are represented as operators defined in rich functional spaces. Converters depend on their initial state as on the parameter. The dynamics can be described by two comparators: Input-status and status-output (Semenov, Kabulova *et al.,* 2015).

The properties of systems with hysteresis are significantly different from those with functional non-linearities (Semenov, Meleshenko *et al.,* 2015). This can be explained by the complexity and non-linear structure of the hysteresis quantizer state space (Semenov, Grachikov *et al.,* 2014). Beyond that point, mathematical models of hysteresis quantizers are generally not smooth, and this increases the difficulty of applying classical methods to analyse the corresponding systems. Economic systems are especially noteworthy among systems with hysteretic properties (Cook, 2003). In the economy, hysteresis phenomena have been observed since the 1950s (Samarskii, 1986). But the data obtained within systems theory was formalized only in the last decade (Cross *et al.,* 2000; 2008a; 2008b;). There are a lot of reasons for that; one of them is the lack of opportunity to conduct an experiment in contrast to technical areas. Recently, the description of hysteresis in the economics is more and more often found in various sources (Macki *et al.*, 1993; Visintin, 1994). Let us cancel some of the results. The article (Cross, 2014) investigates the influence of the hysteretic component on the natural growth of unemployment. The growth of unemployment is simulated by the Preisach model. It is known from economic theory that market mechanisms admit equilibrium (Caporale *et al.,* 2016) in case when demand and supply are equal, thus making it possible to allocate resources efficiently.

However, it is possible to give examples of crises (Kindleberger and Aliber, 2011), when equilibrium prices turned out to be unstable. This fact can be explained by non-compliance of market mechanisms in technological structure of the economics. The reason for the recovery from the crisis was, as a rule, a structural change in the economics.

Insufficient study of the issue on the simultaneous impact of the technological structure of the economics and the pattern of consumer demand on the degree of market mechanisms stability should be noted. In the paper (Cross *et al.,* 2013), the hypothesis that mathematical modelling and numerical results obtained at describing financial markets (Prudnikov *et al.,* 2010) correctly reflect the essence of the processes occurring only through the use of hysteresis quantizers, was proved. One more result relates to the study of the dependency of the economic decline and the potential output (Cross *et al.,* 2012). Unlike the main macroeconomic models, this work suggested a pattern of behaviour that describes economic decline by analogy to the behaviour of water flows in porous media. The following paper describes the hysteretic features applicable to currency markets (Evans, 2011).

Analysing the data from the paper, it may be concluded that the only way to study and analyse economic systems with hysteretic properties is mathematical modelling of these systems (Cross and Lang, 2011). But despite this, there are not many papers, where hysteretic mechanism of pricing would have been properly described at a mathematical level, as opposed to the works where the algorithm is described at the heuristic level. The hysteretic nature of demand depending on the price and purchasing power ratio is already well-known (Kuczmann, 2010).

However, there are no mathematical models (Mayergoyz, 2003) to describe this ratio. The relevance of the article is determined by the fact that there is a number of unsolved problems in the subject of the study, namely, it is necessary to find solutions to the problem, providing hysteretic behaviour of agents associated with the optimization of the production-price strategy (Palley, 2016), as well as of the problems of optimal functioning of companies producing goods, upon condition of hysteretic pricing and competition.

3. Methodology

The purpose of the paper is to develop analytical and numerical methods of analysing the models of economic systems and processes that include hysteretic properties. Several economic problems are considered in the course of the paper, taking into account hysteretic non-linearities, such as:

1. Pricing models, both discrete and continuous, are considered, provided that the behaviour of the economic agents includes hysteretic features.
2. Numerical and analytical methods for optimizing production activities in the context of hysteretic pricing are developed.
3. Functioning of competing production companies in the context of hysteretic behaviour of economic agents is optimized.
4. Multi-objective problem to optimize production, marketing and storage of the goods at the availability of mediators is considered.

The developed models and methods of analysing economic systems with hysteretic properties can be used to improve the adequacy of the formal mathematical description of the respective systems, which is the basis for more accurate and adequate forecasts. Namely, in the context of hysteretic pricing, the algorithms for optimizing activity of production will allow the optimal price and production strategies to be generated from the point of view of maximizing profits (Batkovskiy et al., 2016).

3.1. Research Methods. Hysteresis Quantizers

Mathematical modelling methods were used in the paper, along with the operator interpretation of hysteresis.Two fundamentally different situations, which arise within the study of hysteresis phenomena, may be highlighted. They can be distinguished based on the purpose of the study. The primary task in the first situation is to create a convenient and simple algorithm to build the output based on the pre-set external influences. Upon that, the inputs typically have a fairly simple structure, such as piecewise-linear structure.

The second task is the main one for the regulatory theory. It arises due to the fact that the studied object is a part of a more complex system and cannot be considered in isolation. Then the interpretation of the hysteretic non-linearity can be presented as follows: an operator or a combination of operators defined in a sufficiently rich functional space. As this case is discussed in the paper, below can be found a description of hysteretic non-linearities defined in the meaning by M.A. Krasnosel'skii, and A.V. Pokrovsky.

3.2.1. Non-ideal relay

Below is a description of the two-step relay with thresholds α and β $(\alpha < \beta)$, the operator states space is a pair of numbers {0.1}. $R[\alpha, \beta, x_0]$ establishes the relationship between the input $u(t) \in C_{[0,T]}$ and the variable output

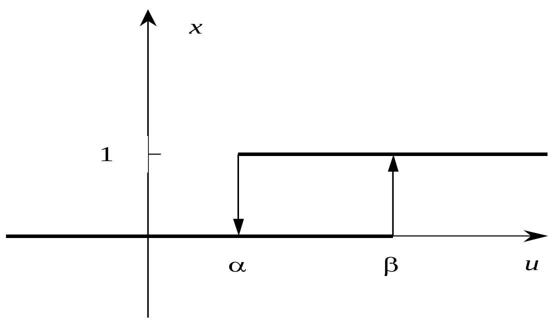
$$x(t) \in \{0,1\} \; x(t) = R\,[\alpha, \beta, x_0]u(t) \tag{1}$$

The initial state of the x_0 converter (1) satisfies the conditions: if $u(0) \le \alpha$, then $x_0 = 0$; if $u(0) \ge \beta$, then $x_0 = 1$; if $\alpha < u(0) < \beta$, then $x_0 = 0$ or $x_0 = 1$. The output $x(t)$ $(0 \le t \le T)$ matches the variable state of converter R and is determined by the correlation:

$$x(t) = \begin{cases} 0, \text{ if } u(t) \le \alpha, \\ 1, \text{ if } u(t) \le \beta, \\ x_0, \text{ if } u(\tau) \in (\alpha, \beta) \text{ for all } \tau \in [0,t], \\ 0, \text{ if } u(t) \in (\alpha, \beta) \text{ provided that} \\ \qquad t_1 \in [0,t) \,\&\, u(t_1) = \alpha \,\&\, u(\tau) \in (\alpha, \beta) \text{ for all } \tau \in (t_1, t], \\ 1, \text{ if } u(t) \in (\alpha, \beta) \text{ provided that} \\ \qquad t_1 \in [0,t) \,\&\, u(t_1) = \beta \,\&\, u(\tau) \in (\alpha, \beta) \text{ for all } \tau \in (t_1, t] \end{cases} \tag{2}$$

Figure 1 illustrates the input/output dependency ratio of the non-ideal relay converter.

Figure 1: *The input/output dependency ratio of the non-ideal relay converter*



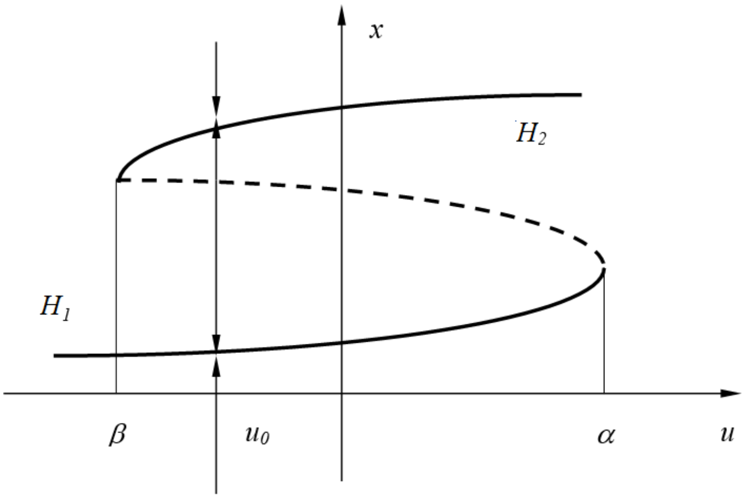
3.2. Mathematical Apparatus of the Study. Relays and Slow Controls

Equation of the following form:

$$f(x,u)=0 \tag{3}$$

describes the curve H, shown in Figure 2.

Figure 2: *S-Converter Level Line*



The curve has one common point with a straight line $u=u_0$, if $u_0<\beta$ or $u_0>\alpha$, and three common points, if $\beta<u_0<\alpha$. The curve H divides the plane into two parts. Let us assume that the function $f(x,u)$ takes positive values at the bottom part, and negative values in the upper part. Let us study the equation

$$\frac{dx}{dt}=f(x,u) \tag{4}$$

with slow control $u=u(t)$.

At $u(t)\equiv u_0$, equation (4), describes the movement of a point on a vertical straight line $u=u_0$; the direction of the movement on one of these straight lines is indicated by the arrows in Figure 2. Equilibrium states are points of intersection of a straight line $u=u_0$ and a curve H; the dashed line represents the part of the curve H, where the equilibrium points will be unsustainable, and the solid line shows asymptotically sustainable points; a solid line consists of graphs H_1 and H_2 of some functions $\gamma_1(u)$ and $\gamma_2(u)$.

Based on this information, the qualitative description of the behaviour of the solution to $x(t)$ equation (4) at slowly changing control $u(t)$ $(t \geq t_0)$ is possible. After a short period of time, the point $\{u(t), x(t)\}$ gets in such a small neighbourhood or points $\{u(t), \gamma_1[u(t)]\}$, or point $\{u(t), \gamma_2[u(t)]\}$ that it can be considered to be coincident with these points. Let us suppose that $u(t_1) < \alpha$ and $x(t_1) \approx \gamma_1[u(t_1)]$; then, at the next values $t \in (t_1, t_2)$, when $u(t) < \alpha$, and point $\{u(t), x(t)\}$ does not leave the small neighbourhood of the curve H_1, the equation $x(t) = \gamma_1[u(t)]$ will be met. If $u(t_2) = \alpha$ and $u(t)$ at the point t_2 increase, then within the time interval where $u(t)$ changes a little, the point $\{u(t), x(t)\}$ gets to the small neighbourhood of the curve H_2, then equation $x(t) = \gamma_2[u(t)]$ is met. From the further discussion, a description of the solution to $x(t)$ is implied, matching the description of the non-ideal relay, the threshold numbers of which are α and β, if H_1 coincides with the semi-straight line $x = 0$ $(u < \alpha)$, and H_2 coincides with the semi-straight line $x = 1$ $(u > \beta)$.

4. Results

4.1. Continuous Hysteretic Pricing Model

In order to describe the pricing, the cobweb model is often used in discrete time systems or equation of the form (5) in models with continuous time:

$$\dot{p} = \gamma(S(p) - D(p)), \tag{5}$$

where *p* is price, *S* (*p*) and *D* (*p*) are demand and supply functions. The only stationary solution to the equation (1) is totally stably and quickly achieved under the traditional assumptions of demand and supply functions. Let us subject the pricing model to the following system:

$$\dot{p} = f(u, p), \tag{6}$$

$$\dot{u} = \gamma(S(p) - D(p)), \tag{7}$$

where the function level line $f(u, p) = 0$ has S-shaped form (see Figure 2)

If the equation (6) is considered in isolation, considering $u(t)$ as a control parameter, then its dynamic allows following qualitative description.

Let us move to the description of the price change dynamics in the system (6), (7). Let us consider that p_2 is the left root of the equation $f(u_2, p) = 0$, and p_1 is the right root of the equation $f(u_1, p) = 0$. The price will be almost unchanged if the solution p^* to the equation *S* (*p*) = *D*(*p*) will satisfy the following condition $p^* \notin [p_1; p_2]$, then the system (6)-(7) will have the only sustainable solution. At $p^* \in [p_1; p_2]$, the system will have a sustainable cycle that corresponds to a cyclical change in prices in the neighbourhood of the unstable equilibrium point. Thus, it is possible to define dependence of prices cyclical change on the selection of the function $f(u, p)$. This article does not address this issue, but let us note that the identification of parameters of this function is possible by standard econometric methods.

4.2. Production Model

Previously, the production model was formed either in terms of production functions or approximated by the linear elements with transfer functions of 1-2 order. In this article, the wording of the production model looks as follows: Let us consider that *U* is the production rate, *z* is the number of products in the possession of the manufacturer. Then the following equation will be introduced to describe the dynamics of changing the product quantity in the possession of the manufacturer:

$$\dot{z} = U - NQ(\frac{S}{p}), \tag{8}$$

where *N* is the number of consumers; *S* is the consumers' income; $Q(\frac{S}{p})$ is the demand function, the specific type of which will be defined below.
V denotes the number of goods in the possession of the consumers, and *k* is the coefficient of propensity to consume. Then the dynamics of the change will be:

$$\dot{V} = NQ(\frac{S}{p}) - kV \tag{9}$$

The demand function will be defined as follows:

$$Q\left(\frac{S}{p}\right) = q - a\frac{p}{S}, \tag{10}$$

Q is the maximum product requirement for one person per unit of time.
As soon as the price rises, the demand function falls and at $p = p_{cr}$ consumers refuse to purchase the goods. Value of $p_{cr} = Sq/a$. The parameter a is the measure of the elasticity of the demand function by price. Therefore, the conclusion can be made that the demand function defined by type (10) is the threshold function (i. e., $Q\,(S/p) = 0$ under $p \geq p_{cr}$) and it has saturation property. The supply function $D(p,z)$ is defined as follows:

$$D(p,z) = z\frac{p}{p_{cr}}. \qquad (11)$$

The manufacturer's profit at the final time interval [0;T] can be determined by the correlation (12), herewith p_0 is the cost per unit, and k is the input coefficient for storing:

$$J(T) = \int_0^T (NpQ\left(\frac{S}{p}\right) - Up_0 - k_1 z)dt. \qquad (12)$$

Let us assume that the maximum production rate limited by technological limitations $0 \leq U \leq U_0$, U_0 is the maximum production rate. Then the problem is reduced to the problem of optimal control: It is necessary to determine the production rate in which the functional (12) reaches the maximum value and the system will be described by differential equations (6)-(9) and algebraically ratios (10), (11). The solution of that problem will be provided through standard methods of optimal control theory. The Hamiltonian function is given by:

$$\begin{aligned} H = NpQ\left(\frac{S}{p}\right) - Up_0 - k_1 z + \lambda_1 f(p,u) + \lambda_2(S(p) - D(p)) + \\ + \lambda_3(U - NQ\left(\frac{S}{p}\right)) + \lambda_4(NQ\left(\frac{S}{p}\right) - kV) \end{aligned} \qquad (13)$$

Due to the Pontryagin maximum principle, the optimal production rate will be determined by the following correlation: $U^* = \arg\max_U H.$

As the Hamiltonian function is linear in U, the optimal production rate will equal to:

$$U^* = \begin{cases} U_0, & if \quad \lambda_3 \geq p_0, \\ 0, & if \quad \lambda_3 < p_0, \end{cases} \qquad (14)$$

thus, the conclusion can be made that the optimal production rate is represented by the relay function. Conjugated variables meet the following equations:

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial p},$$
$$\dot{\lambda}_2 = -\frac{\partial H}{\partial u},$$
$$\dot{\lambda}_3 = -\frac{\partial H}{\partial z},\qquad (15)$$
$$\dot{\lambda}_4 = -\frac{\partial H}{\partial V}.$$

The conjugated variables are superposed by the boundary conditions, as the problem under consideration is problem with two variable end-points:

$$\lambda_i(T) = 0 \quad (i = 1,2,3,4), \qquad (16)$$

The system consisting of equations, (6)-(11), (14), (15) is closed. The following figures show the results of the numerical analysis. The following parameters were considered: $\gamma = 2$, $k = 0{,}5$, $k_1 = 0{,}6$, $S = 5$, $q = 10$, $N = 10$, $U_0 = 3$, $T = 20$.

Figure 3: *Projection of a phase space to a plane* p, V

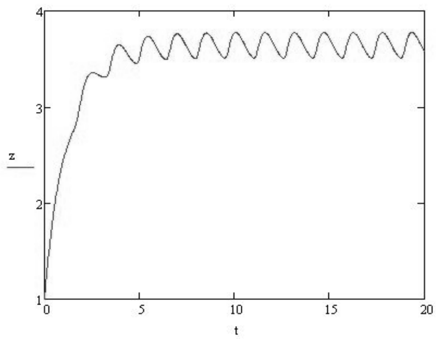


Figure 4: *Product quantity in the possession of the manufacturer depending on time*

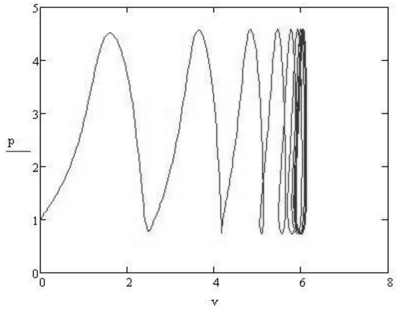
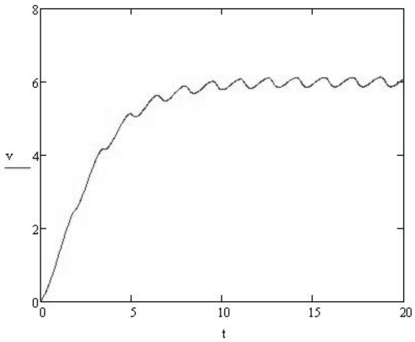


Figure 5: *Product quantity in the possession of the consumer depending on time*



It is clear from the graphs that the phase variables in the steady-state operating conditions perform weak fluctuations in the neighbourhood of the unstable equilibrium point. Hysteretic price behaviour is the reason for that.

4.3. Production Model of Pricing Based on Hysteretic Factors

Mechanisms of control and modelling of the production process and goods consumption is an important objective of the economic strategy (Batkovskiy et al., 2015). One of the important areas in the study of economic processes is the study of the economic agents' behaviour in the neighbourhood of the equilibrium point. Let us note that that the status of the economic system at some point in time depends both on the fixed parameter values and on their values at previous points of time. This behaviour is characteristic of the hysteresis quantizers.

Let us introduce $R[\alpha,\beta,x_0]$ to designate a two-position relay with threshold numbers of α and β.

Below is the description of the private class of the Preisach converters for which the condition $D_{\alpha,\beta} \equiv \{\alpha, \beta : \alpha < \beta\}$ is true. Then the measure μ should be determined on the half-plane $P_{\alpha,\beta}$ by the next equation:

$$d\mu = \lambda(\alpha, \beta) d\alpha d\beta \tag{17}$$

Let us take that ψ is the confined functions class set on a non-negative semi axis and satisfy the Lipchitz condition with a coefficient equal to one. Also it is necessary to introduce the set Ω_ψ of scalar-valued functions $\omega(\alpha, \beta)$ defined on the half-plane $P_{\alpha,\beta} \equiv \{\alpha, \beta : \alpha < \beta\}$:

$$\omega(\alpha, \beta) = \begin{cases} 0, & if \;\; \alpha + \beta > \psi(\beta - \alpha), \\ 1, & if \;\; \alpha + \beta \le \psi(\beta - \alpha), \end{cases} \tag{18}$$

$\psi(\nu) \in \psi$, set Ω_ψ is the space of probable states of the Preisach converter.

Let us specify an arbitrary element $\omega_0(\alpha, \beta) \in \Omega_\psi$ and accept that for the Preisach converter (H, ξ), with the initial state $\omega_0(\alpha, \beta)$. all continuous inputs $u(t), t \ge 0$ satisfying the equation $u(0) = \psi_0(0)$, where $\omega_0(\alpha, \beta)$ and $\psi_0(\nu)$, are related by the correlation (18).
The input is defined by the correlation:

$$\omega(\alpha(\gamma), \beta(\gamma), t) = H\left[\omega_0(\gamma)\right]u(t) = R\left[\omega_0(\alpha, \beta, \gamma), \alpha(\gamma), \beta(\gamma)\right]u(t) , \tag{19}$$

where γ is the parameter, $\gamma \in P_{\alpha,\beta}$.

And the output is:

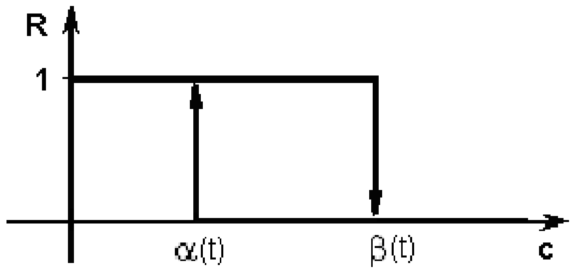
$$\xi(t) = \int\limits_{\alpha \le \beta} \omega(\alpha, \beta, t) d\mu = \mu(\{\alpha, \beta\} : R[\omega_0(\alpha, \beta), \alpha, \beta]u(t) = 1) \tag{20}$$

Let us use a modification of the converters described above to compose the model of consumer demand. Let us accept that the demand function *P*(*t*) at a moment of time *t* depends only on price *c*(*t*). Then the customer relationship to the each product is determined by the function $R(c(t))$:

$$R(c(t)) = \begin{cases} 1, & if \quad c(t) \le \alpha(t), \\ 0, & if \quad c(t) \ge \beta(t), \\ 1 \quad or \quad 0, & if \quad \alpha(t) < c(t) < \beta(t). \end{cases} \tag{21}$$

When the product is purchased, the function $R(c(t))$ equals to one, otherwise it equals to zero. Function $R(c(t))$ is defined as the output of a converter $R[\alpha(t), \beta(t), R_0]$, which is the equivalent of non-ideal relay with the inversion of the threshold numbers α, β to the input of which signal $c(t)(t \ge 0)$ is set. Figure 6 shows the relationship between the input and the output.

Figure 6: *Input/output correlations of the converter (21)*



It is also possible to take into account the customer's relationship to the product that changes over time, since the threshold numbers $\alpha(t)$ and $\beta(t)$ are time dependent. If γ_i is the rate of purchase of i th consumer $(i = 1, 2 \ldots n)$, then the system of n consumers takes the following form:

$$P(c(t)) = \sum \gamma_i R[\alpha_i(t), \beta_i(t), R_{0i}]c(t) \tag{22}$$

In a continuous case, the sales function equals to the Preisach converter with inversion of ones and zeros::

$$P(c(t)) = \int_{\alpha \le \beta} \omega(\alpha(t), \beta(t), t) d\mu(t) \tag{23}$$

Where

$$\omega(\alpha(t), \beta(t), t) = H[\omega_0(\alpha, \beta)]c(t) = R[\alpha(\gamma, t), \beta(\gamma, t), R_0(\gamma)]c(t), \gamma \in P_{\alpha, \beta} \tag{24}$$

The continuous analogue of the non-ideal relay takes into account possible changes in the consumer's individual relationship to the product. In the model, this is reflected by dependency of measure μ on time.

Equilibrium in market mechanisms is possible in cases where demand and supply are equal and efficient allocation of resources is carried out. The solution to that problem is based on the consumer's demand model, which takes into account its inertia and the possibility of structural changes. The consideration involves exclusively single goods market, the main feature of which is that the purchase and sale of goods occurs directly between consumers and manufacturers. At the moment of time n, the product is sold at a single price of c_n; the consumer's behaviour is described by the demand function P (c) whose analytical view is defined by the converter; the manufacturer's behaviour can be described by supply function g (c), below one can find its analytical form. It should be noted that the change in the functions of supply and demand occurs over the time much longer than the time required for price to change.

Based on the assumption that manufacturers offer the product at the "previous" price c_{n-1} and sell it at "present" c_n. However, the consumers are ready to buy the product at the previous price P (c_{n-1}) and ready to exactly the previous price c_{n-1} for it. Then the equilibrium of demand and supply is determined by the following correlation:

$$c_n q(c_{n-1}) = c_{n-1} q(c_{n-1}) \tag{25}$$

Based on the paper by G.S. Pospelov, let us define the following function:

$$g(p_{n-1}, t) = \int_{\nu}^{c_{n-1}/s} \xi(\lambda, t) d\lambda, \tag{26}$$

t corresponds to the change of production capacity, in other words "slow time, and n is discrete " fast time", qualifies for the processes connected with price changing; $\xi = \xi(\lambda, t)$ is the smooth function of capacity distribution according to the production technologies.

In operating principle of the exponential growth of new technologies with the rate of γ:

$$I(t) = M(\tau_0)(\gamma + \mu)e^{\gamma(t-\tau_0)} \tag{27}$$

With the initial distribution of capacities $\xi(\lambda, \tau_0)$ and the supply function $g(p, t)$ at $\tau_0 \le t \le \tau$.

Let us show that:

$$g\left(c_{n-1}, t\right) = M\left(t\right)\left[1 - \left(\frac{s\nu}{c_{n-1}}\right)^{(\gamma+\mu)/\mu}\right] \tag{28}$$

Let us define as follows:

$$x_n = \frac{s\nu}{p_n}, \quad \alpha = \frac{\gamma+\mu}{\mu} \geq 1, \quad A(t) = \frac{M(t)}{P(c)}, \tag{29}$$

Let us define $P(c)$ as the output from the converter at the time t = 1, to the input of which the signal is set:

$$\phi(t) = tc_n + (1-t)c_{n-1}, \tag{30}$$

in other words, this is a linear function that connects the previous and current values of the price.
Therefore:

$$x_n = A(t)x_{n-1}(1 - x_{n-1}^{\alpha}) \tag{31}$$

Based on the assumption that the best technology is not unprofitable $x_n = s\nu / c_n \leq 1$. Then we get $0 \leq x_n \leq 1$. In order to convert a segment $[0, 1]$ into itself through displaying (31), the following is necessary and sufficient:

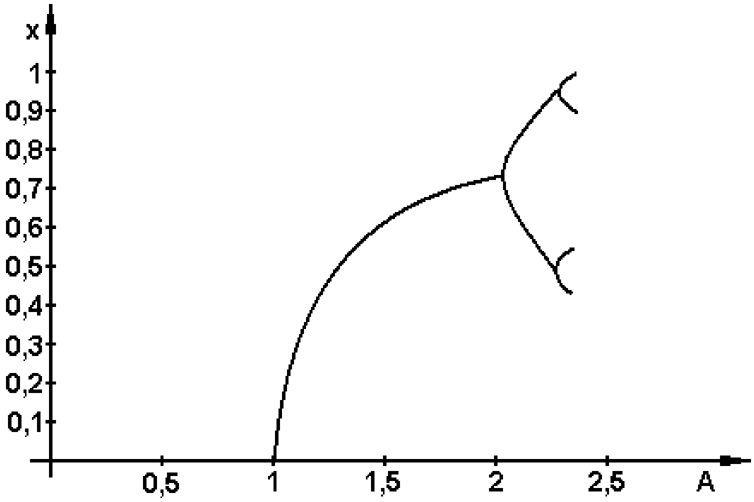
$$0 \leq A(t) \leq \frac{(1+\alpha)^{(1+\alpha)/\alpha}}{\alpha} = A_M(\alpha) \tag{32}$$

Discrete dynamic system in which "slow time" t is a parameter determined by the mapping (31). The model of pricing described above $x_{n+1} = f(x_n, A, \alpha)$, where $f(x, A, \alpha) = Ax(1-x^{\alpha})$ is capable of define determination of an univalently infinite path $x_0, x_1, \ldots, x_n, \ldots$ by the given initial condition x_0. Considering the time-splitting hypothesis, let us study asymptotic (at $n \to \infty$) price behaviour.
$x = 0$ is the static point of the dynamic system (31) at all $A \geq 0, \quad \alpha \geq 1$. At the different parameter values $A \geq 1, \quad \alpha \geq 1$, there is another fixed point $x_p(A, \alpha) = (1 - 1/A)^{1/\alpha}$, which corresponds to the equilibrium price. The only path where the manufacturer's and consumer's price forecasts coincide with the

implementation, and production is consistent with demand, is defined by the exact $x_p(A, \alpha)$.

Let us analyse the correlation of increase in parameter A and asymptotic behaviour of the dynamic paths of system (31). When $0 \leq A < 1$, then regardless of the initial condition x_0, the path of the system (31) tends to 0. Bifurcation occurs at A = 1, as a result, the fixed point $x = 0$ becomes unstable and a stable (at A close to 1) point $x_p(A, \alpha)$ appears (Paraev, 2000). Figure 7 illustrates the bifurcation tree, namely it shows how the attractor depends on the parameter A.

Figure 7: *Bifurcation Tree*



Analysing Figure 7, let us note that while increasing parameter $A = M / P$, it becomes more difficult to determine price changes dynamics; and this is the main obstacle to business activity of economic agents (Mishin, 2014).

The paper establishes the dependence of the effective use by the managing body of economic resources on the parameters of the stability of market relations; in the presented model they are determined by the parameters $A_1(\alpha), A_2(\alpha), \ldots, A_\infty(\alpha)$. With the increase in product offerings, the range of stability of market mechanisms is delining, consistent with the economists' view of the overheated economy (Shananin, 1986).

4.4. Multi-objective problem on producing, consuming, and selling the product having a mediator

Summarizing the results obtained earlier; let us assume the interaction of the manufacturer and the mediator, who in turn, sells the goods to the final consumer. In these circumstances, the problem is multi-objective, as the target of both the consumer and the mediator is maximizing their income.

The main parameters, when considering the multi-objective problem of producing, consuming, and selling the product at the availability of a mediator, are as follows: Z_1 (t), Z_2 (t), Z_3 (t) is the quantity of product available in the possession of the manufacturer, mediator and consumer. Let us define the production rate as U (t). P_1 (t), P_2 (t) is the amount of sales in a unit of time by the manufacturer and the mediator. k_1 is the consumption factor, k_2 is the input factor for storing a unit of product of the manufacturer, k_3 is the input factor for storing a unit of product of the mediator, c_1 (t), c_2 (t) is the unit price of the manufacturer and the mediator. Changes to the entered values can be described by the following system of equations:

$$\dot{Z}_1 = U - P_1, \tag{33}$$

$$\dot{Z}_2 = P_1 - P_2, \tag{34}$$

$$\dot{Z}_3 = P_2 - kZ_3, \tag{35}$$

$$P_1 = Z_1(a_1 - b_1c_1), \tag{36}$$

$$P_2 = Z_2(a_2 - b_2c_2) \tag{37}$$

where a_1, b_1, a_2, b_2 are positive constants.

Function determining profit:

$$J_1 = \int_0^T (c_1P_1 - k_1U - k_2Z_1)dt \to \max, \tag{38}$$

$$J_2 = \int_0^T (c_2P_2 - c_1P_1 - k_3Z_2)dt \to \max \tag{39}$$

Let us express weighted income J (t) taking into account the introduced denotations:

$$J(\alpha) = \alpha J_1 + (1-\alpha)J_2, \tag{40}$$

$$J(\alpha) = \int_0^T (2\alpha c_1 P_1 - c_1 P_1 - k_1 U - k_2 Z_1 - k_3 Z_2 + c_2 P_2 - \\ - \alpha c_2 P_2 + \alpha k_3 Z_2) dt \to \max \tag{41}$$

Applying the Pontryagin maximum principle and making up the Hamilton function, we get:

$$H = 2\alpha c_1 P_1 - c_1 P_1 - k_1 U - k_2 Z_1 - k_3 Z_2 + c_2 P_2 - \alpha c_2 P_2 + \\ + \alpha k_3 Z_2 - \lambda_1 (U - P_1) - \lambda_2 (P_1 - P_2) - \lambda_3 (P_2 - kZ_3), \tag{42}$$

$$U^* = \begin{cases} 0, \lambda_1 + k_1 \le 0, \\ U_0, \lambda_1 + k_1 > 0. \end{cases} \tag{43}$$

The equation for the conjugated variables is the following:

$$\dot{\lambda}_3 = 0, \ \dot{\lambda}_2 = c_2 (1-\alpha) - \frac{k_3 (1-\alpha)}{a_2 - b_2 c_2}, \tag{44}$$

$$\dot{\lambda}_1 = c_2 (1-\alpha) - \frac{k_3 (1-\alpha)}{a_2 - b_2 c_2} + c_1 (1-2\alpha) + \frac{\alpha k_2}{a_1 - b_1 c_1}, \tag{45}$$

$$\lambda_1 (T) = \lambda_2 (T) = \lambda_3 (T) = 0 \tag{46}$$

Out of (45) maximizing for c_1 and c_2:

$$c_1^* = \frac{\lambda_2 - \lambda_1}{2(2\alpha - 1)} + \frac{a_1}{2b_1}, \tag{47}$$

$$c_2^* = \frac{\lambda_3 - \lambda_2}{2(1-\alpha)} + \frac{a_2}{2b_2} \tag{48}$$

At $\alpha = 0.5$, the specific solution looks as follows:

$$J_1 = \int_0^T (c_1 p_1 - k_1 u - k_2 z_1) dt \to \max, \tag{49}$$

$$J_2 = \int_0^T (c_2 p_2 - c_1 p_1 - k_3 z_2) dt \to \max \tag{50}$$

Let us define weighted income *J* (*t*):

$$J = J_2 + J_1 = \int_0^T (c_2 p_2 - k_1 u - k_2 z_1 - k_3 z_2) dt \to \max \tag{51}$$

The Hamiltonian function is written as follows:

$$H = c_2 p_2 - k_1 u - k_2 z_1 - k_3 z_2 - \lambda_1 (u - p_1) - \lambda_2 (p_1 - p_2) - \lambda_3 (p_2 - kz_3) \tag{52}$$

Then, the solution is as follows:

$$\dot{\lambda}_3 = 0 \tag{53}$$

$$\dot{\lambda}_2 = c_2 - \frac{k_3}{a_2 - b_2 c_2} \tag{54}$$

$$\dot{\lambda}_1 = c_2 - \frac{k_3}{a_2 - b_2 c_2} + \frac{k_2}{a_1 - b_1 c_1} \tag{55}$$

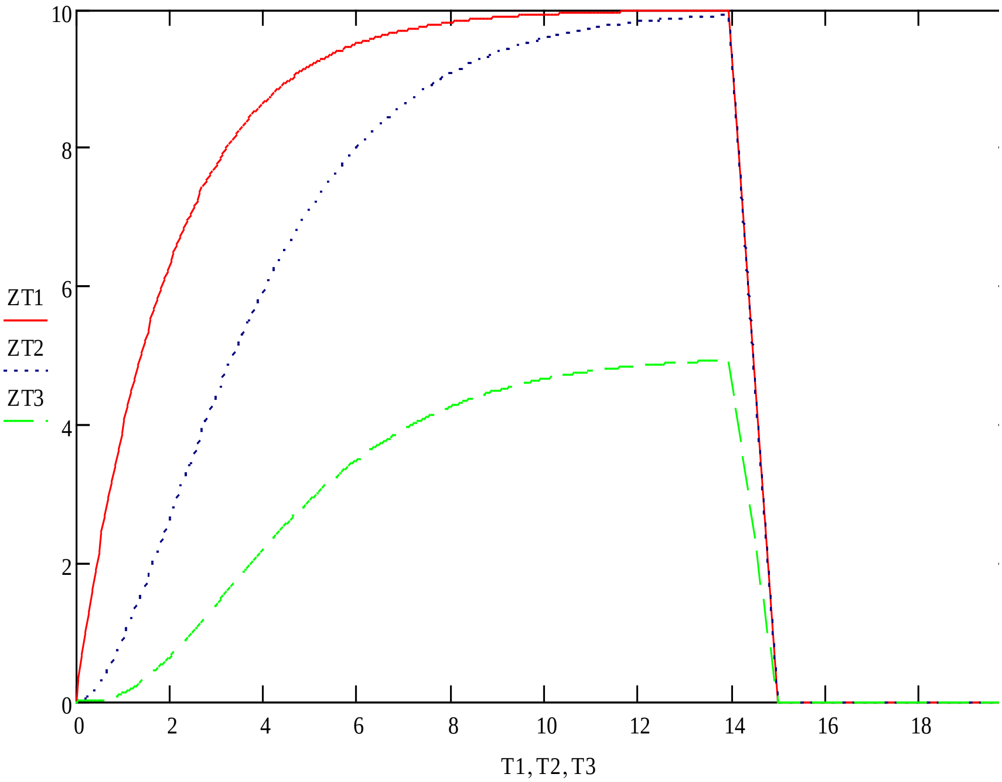
$$U^* = \begin{cases} 0, \lambda_1 + k_1 \geq 0, \\ U_0, \lambda_1 + k_1 < 0 \end{cases} \tag{56}$$

$$c_1^* = \begin{cases} 0, \lambda_1 - \lambda_2 < 0, \\ c_{1\max}^*, \lambda_1 - \lambda_2 \geq 0 \end{cases} \tag{57}$$

$$c_2^* = \frac{\lambda_3 - \lambda_2}{2} + \frac{a_2}{2b_2} \tag{58}$$

Numerical solution of the system (33)-(39), (43)-(48):

Figure 8: Dependency z_1, z_2, z_3 on t



Let us consider the functional:

$$J(\alpha) = \alpha J_1 + (1-\alpha) J_2 \to \min \tag{59}$$

$Z^*(t,\alpha)$, $c^*(t,\alpha)$ optimal solution of (54) - (59), at the conditions of smooth dependence of the functional on the parameter α:

$$J^* = \int_0^{t_f} \alpha f_1(Z^*(\alpha,t), c^*(\alpha,t)) + (1-\alpha) f_2(z^*(\alpha,t), c^*(\alpha,t)) \tag{60}$$

From the condition $\dfrac{dJ}{d\alpha} = 0$:

$$\frac{dJ}{d\alpha}=\int_0^{t_f}(f_1-f_2)+\alpha(\frac{df_1}{dZ}*\dot{Z}_\alpha{}^*+\frac{df_1}{dc}*\dot{Z}_\alpha{}^*)+(1-\alpha)\left[f_2\frac{df}{dZ}*Z_\alpha^*+\frac{df_2}{dc}*c\right]\partial t=$$
$$=\int_0^{t_f}\left[(f_1-f_2)+\dot{Z}_\alpha^*(\alpha\frac{df_1}{dZ}+(1-\alpha)\frac{df_2}{dZ})+\dot{c}_\alpha^*(\alpha\frac{df_1}{dc}+(1-\alpha)\frac{df_2}{dc})\right]dt \tag{61}$$

Also:

$$H=\alpha f_1+(1-\alpha)f_2+\lambda f \tag{62}$$

If the paths are optimal, then:

$$\frac{dH}{dc}=0;\ \dot{\lambda}=-\frac{dH}{dZ}, \tag{63}$$

Then:

$$\alpha\dot{f}_1c+(1-\alpha)\dot{f}_2+\lambda\dot{f}_c=0, \tag{64}$$
$$\dot{\lambda}=-(\alpha\dot{f}_{1Z}+(1-\alpha)\dot{f}_{2Z}+\lambda\dot{f}_Z), \tag{65}$$
$$\alpha\dot{f}_{1c}+(1-\alpha)\dot{f}_{2c}=-\lambda\dot{f}_c, \tag{66}$$
$$\alpha\dot{f}_{1Z}+(1-\alpha)\dot{f}_{2Z}=-\dot{\lambda}-\lambda\dot{f}_Z \tag{67}$$

By plugging (66) and (67) into (61) the following is obtained:

$$\frac{dJ^*}{d\alpha}=\int_0^{t_f}\left[(f_1-f_2)+\dot{Z}_\alpha^*(-\dot{\lambda}-\lambda\dot{f}_c)+\dot{c}_\alpha^*(-\lambda\dot{f}_c\right]dt=$$
$$=\int_0^{f_t}\left[(f_1-f_2)-\dot{Z}_\alpha^*\dot{\lambda}-\lambda(\dot{f}_Z*\dot{Z}_\alpha^*+\dot{f}_c*\dot{c}_\alpha^*)\right]dt= \tag{68}$$
$$=\int_0^{t_f}\left[(f_1-f_2)-\dot{Z}_\alpha^*\dot{\lambda}-\lambda\frac{df}{d\alpha}\right]dt$$

Let us convert the second summand (68):

$$-\int_0^{t_f} \dot{Z}_\alpha^* \dot{\lambda}\, dt = -\int_0^{t_f} Z^* d\lambda = -\dot{Z}_\alpha^* \lambda \,\Big|_0^{t_f} + \int_0^{t_f} \lambda \frac{d}{dt} \dot{Z}_\alpha^* dt =$$

$$= -Z^* \lambda(t_f) + \dot{Z}_\alpha^* \lambda(0) + \int_0^{t_f} \lambda \dot{Z}_\alpha^* dt = \dot{Z}_\alpha^*(0)\lambda(0) + \int_0^{t_f} \lambda \frac{d}{d\alpha} f dt \qquad (69)$$

$$\frac{dJ}{d\alpha} = \int_0^{t_f} \left[(f_1 - f_2) + \lambda \frac{df}{d\alpha} - \lambda \frac{df}{d\alpha} \right] dt + (\dot{Z}_\alpha^*(0), \lambda(0))$$

At $\frac{dJ}{dt} = 0$, the equation is satisfied:

$$\int_0^{t_f} (f_1(Z^*, c^*) - f_2(Z^*, c^*)) dt = (-\dot{Z}_\alpha^*(0), \lambda(0)) \qquad (70)$$

In the course of the study, a mathematical model was constructed to solve the multi-objective problem of production, storage and selling of products, subject to competition criteria based on the conditions of the interests of the mediators and trade organizations. The numerical solution was also received for the stated problem. It was possible to establish the dependence of the weighted profit functional on parameters for a multi-objective problem. The optimality condition for a weighted criterion from a parameter for a multi-objective problem was obtained as a result of the study. The proposed numerical algorithm allows to find the optimal solution for the optimal production, consumption, and selling of the products when a mediator is available.

4. Conclusion

The paper discusses various aspects of the dynamics of economic systems, taking into account the hysteretic properties. As shown by the results of the modelling, taking into account hysteresis properties significantly complicates the analysis of various economic systems. This is because the output of the hysteretic link depends not only on the value of the input at the certain moment of time, but also on the state of the hysteresis quantizer and the value of the input at the previous moment of time. Economic systems, as shown by numerous studies, have similar characteristics, so the use of the means of hysteretic converters for the analysis of their dynamics is appropriate and justified.

One of the results of the work is the development of a method for studying the dynamics of pricing in conditions of hysteretic behaviour of economic agents. The proposed model at the formal level takes into account the inertia of consumer demand, which increases the adequacy of mathematical modelling of consumer behaviour. Traditionally, the behaviour of individual consumers (as well as their peer groups) was modelled by a utility function that cannot account for the propensity of consumers to buy "known" goods.

The next result concerns the development of a model of an optimal production strategy under conditions of hysteretic pricing. In particular, based on the L. S. Pontryagin maximum principle, we obtain the conditions that ensure the maximization of manufacturer's profits in the single goods markets. The obtained result can serve as the basis for the development of optimal (in terms of achieving maximum financial performance at a finite time interval) production and price strategy.

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